

Comments, suggestions, corrections, and references are most welcomed!

ON THE ORDERS OF GENERATORS OF CAPABLE p -GROUPS

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ABSTRACT. A group is called capable if it is a central factor group. For each prime p and positive integer c , we prove the existence of a capable p -group of class c minimally generated by an element of order p and an element of order $p^{1+\lfloor \frac{c-1}{p-1} \rfloor}$. This is best possible.

1. INTRODUCTION

Recall that a group G is said to be **capable** if and only if G is isomorphic to $K/Z(K)$ for some group K , where $Z(K)$ is the center of K . There are groups which are not capable (nontrivial cyclic groups being a well-known example), so capability places restrictions on the structure of a group; see for example [3, 4]. As noted by P. Hall in his landmark paper on the classification of p -groups [2], the question of which p -groups are capable is interesting and plays an important role in their classification.

P. Hall observed that if G is a capable p -group of class c , with $c < p$, and $\{x_1, \dots, x_n\}$ is a minimal set of generators with $o(x_1) \leq o(x_2) \leq \dots \leq o(x_n)$ (where $o(g)$ denotes the order of the element g), then $n > 1$ and $o(x_{n-1}) = o(x_n)$.

In [5] we used commutator calculus to derive a similar necessary condition after dropping the hypothesis $c < p$: if G is a capable p -group of class $c > 0$, minimally generated by $\{x_1, \dots, x_n\}$, where $o(x_1) \leq \dots \leq o(x_n)$, then we must have $n > 1$ and letting $o(x_{n-1}) = p^a$ and $o(x_n) = p^b$, then a and b must satisfy

$$(1.1) \quad b \leq a + \left\lfloor \frac{c-1}{p-1} \right\rfloor,$$

where $\lfloor x \rfloor$ is the greatest integer less than or equal to x (Theorem 3.19 in [5]). The dihedral group of order 2^{c+1} shows that (1.1) is best possible when $p = 2$. The purpose of this note is to show that the inequality is best possible for all primes p , thus answering in the affirmative Question 3.22 in [5].

Notation will be standard; all groups will be written multiplicatively, and we will denote the identity by e . We use the convention that the commutator of two elements x and y is $[x, y] = x^{-1}y^{-1}xy$. The lower central series of G is defined recursively by letting $G_1 = G$, and $G_{n+1} = [G_n, G]$; we say G is nilpotent of class (at most) c if and only if $G_{c+1} = \{e\}$. It is well known that if G is of class exactly c , then $G_c \subset Z(G)$, and $G/Z(G)$ is nilpotent of class exactly $c - 1$.

We let C_n denote the cyclic group of order n , and \mathbf{Z} the infinite cyclic group, both written multiplicatively.

2. THE CASE $c = 1 + (r - 1)(p - 1)$

The construction in this section is based on the example given by Easterfield in Section 4 of [1].

Let p be a prime, r a positive integer. We construct a p -group K of class $c + 1 = 2 + (r - 1)(p - 1)$, minimally generated by an element y of order p , and an element x_0 of order p^r . We will show that the images of y and x_0 have the same order in $K/Z(K)$, thus exhibiting a capable group of class $c = 1 + (r - 1)(p - 1)$, minimally generated by an element of order p and one of order $p^{1 + \lfloor \frac{c-1}{p-1} \rfloor}$.

Let H be the abelian group

$$H = C_{p^r} \times C_{p^r} \times \underbrace{C_{p^{r-1}} \times \cdots \times C_{p^{r-1}}}_{p-2 \text{ factors}}.$$

Denote the generators of the cyclic factors of H by x_0, x_1, \dots, x_{p-1} , respectively. If $r = 1$, then x_1, \dots, x_{p-1} are trivial. Let y generate a cyclic group of order p , and let y act on H by $y^{-1}x_i y = x_i x_{i+1}$ for $0 \leq i \leq p-2$ (so $[x_i, y] = x_{i+1}$), and

$$y^{-1}x_{p-1}y = x_1^{-\binom{p}{1}} x_2^{-\binom{p}{2}} \cdots x_{p-2}^{-\binom{p}{p-2}} x_{p-1}^{1-\binom{p}{p-1}};$$

as usual, $\binom{n}{k}$ is the binomial coefficient n choose k . Let $K = H \rtimes \langle y \rangle$.

Remark 2.1. The group constructed by Easterfield is the subgroup of K generated by y and x_1, \dots, x_r . We can also realize K as the semidirect product of this subgroup by $\langle x_0 \rangle$, letting x_0 act on y by $x_0^{-1}y x_0 = y x_1^{-1}$, and act trivially on the x_i .

Note that K is metabelian of class exactly $2 + (r - 1)(p - 1)$. To verify the class, note that $[K, K] = \langle x_1, \dots, x_{p-1} \rangle$. We then have:

$$\begin{aligned} K_3 &= \langle x_1^p, x_2, \dots, x_{p-1} \rangle; \\ K_4 &= \langle x_1^p, x_2^p, x_3, \dots, x_{p-1} \rangle; \\ &\vdots \\ K_{2+(p-1)} &= \langle x_1^p, x_2^p, \dots, x_{p-1}^p \rangle; \\ K_{2+(p-1)+1} &= \langle x_1^{p^2}, x_2^p, \dots, x_{p-1}^p \rangle; \\ &\vdots \\ K_{2+k(p-1)} &= \langle x_1^{p^k}, x_2^{p^k}, \dots, x_{p-1}^{p^k} \rangle; \\ &\vdots \\ K_{2+(r-1)(p-1)} &= \langle x_1^{p^{r-1}}, x_2^{p^{r-1}}, \dots, x_{p-1}^{p^{r-1}} \rangle = \langle x_1^{p^{r-1}} \rangle. \end{aligned}$$

Finally, note that $x_1^{p^{r-1}}$ is central: $y^{-1}x_1^{p^{r-1}}y = (x_1 x_2)^{p^{r-1}} = x_1^{p^{r-1}}$. Therefore K is of class exactly $2 + (r - 1)(p - 1)$.

The group $G = K/Z(K)$ will therefore be of class $1 + (r - 1)(p - 1)$, minimally generated by $yZ(K)$ and $x_0Z(K)$. The order of $yZ(K)$ is of course equal to p . As for $x_0Z(K)$, note that no nontrivial power of x_0 is central: if x_0^k is central, then

$$x_0^k = y^{-1}x_0^k y = (y^{-1}x_0 y)^k = (x_0 x_1)^k = x_0^k x_1^k;$$

therefore $x_1^k = e$, which implies that $p^r | k$, so $x_0^k = e$. Therefore, the order of $x_0 Z(K)$ is p^r . Thus, G is a capable group of class c , with $c = 1 + (r-1)(p-1)$, minimally generated by an element of order p and an element of order $p^r = p^{1+\lfloor \frac{c-1}{p-1} \rfloor}$.

We note the following fact about K , which we will use in the following section:

Lemma 2.2. *Let p be any prime, and let r be an arbitrary positive integer. There exists a group K of class $2 + (r-1)(p-1)$, generated by elements y and x_0 of orders p and p^r , respectively, such that $x_0^{p^{r-1}}$ does not commute with y .*

3. GENERAL CASE

Again, let p be a prime, and let $c > 1$ be an arbitrary integer. We want to exhibit a capable group G of class exactly c , generated by an element of order p and an element of order $p^{1+\lfloor \frac{c-1}{p-1} \rfloor}$.

Our construction in this section will be based on the nilpotent product of groups; we specialize the definition to the case we are interested in:

Definition 3.1. Let A_1, \dots, A_n be cyclic groups, and let $c > 0$. The c -nilpotent product of the A_i , denoted $A_1 \amalg^{\mathfrak{N}_c} \dots \amalg^{\mathfrak{N}_c} A_n$ is defined to be the group F/F_{c+1} , where F is the free product of the A_i , $F = A_1 * \dots * A_n$, and F_{c+1} is the $(c+1)$ -st term of the lower central series of F .

It is easy to verify that the c -nilpotent product of the A_i is of class exactly c , and that it is their coproduct (in the sense of category theory) in the variety \mathfrak{N}_c of all nilpotent groups of class at most c . The 1-nilpotent product is simply the direct sum of the A_i .

Note that if G is the c -nilpotent product of the A_i , then G/G_{k+1} is the k -nilpotent product of the A_i for all k , $1 \leq k \leq c$.

We consider $\mathcal{G} = C_p \amalg^{\mathfrak{N}_{c+1}} \mathbf{Z}$, the $(c+1)$ -nilpotent product of a cyclic group of order p and the infinite cyclic group. Denote the generator of the finite cyclic group by a , and the generator of the infinite cyclic group by z . Let $G = \mathcal{G}/Z(\mathcal{G})$. Then G is capable of class c . We want to show that $zZ(\mathcal{G})$ has the required order.

Proposition 3.2. *Let a generate C_p and z generate the infinite cyclic group \mathbf{Z} . If $\mathcal{G} = C_p \amalg^{\mathfrak{N}_{c+1}} \mathbf{Z}$, then*

$$Z(\mathcal{G}) \cap \langle z \rangle = \langle z^{p^{1+\lfloor \frac{c-1}{p-1} \rfloor}} \rangle.$$

Proof. The fact that $z^{p^{1+\lfloor \frac{c-1}{p-1} \rfloor}}$ is central follows from Theorem 3.16 in [5], so we just need to prove the other inclusion. We proceed by induction on c . The claim is true if $c = 1$ since the commutator bracket is bilinear in a group of class two. Assume the inclusion holds for $c-1$, with $c > 1$. Note that $\langle z \rangle \cap \mathcal{G}_2 = \{e\}$.

Consider $\mathcal{G}/\mathcal{G}_{c+1}$; this is the c -nilpotent product of C_p and \mathbf{Z} , so by the induction hypothesis, the intersection of the center and the subgroup generated by z is generated by the $p^{1+\lfloor \frac{c-2}{p-1} \rfloor}$ -st power of z . Since the center of \mathcal{G} is contained in the pullback of the center of $\mathcal{G}/\mathcal{G}_{c+1}$, we deduce that the smallest power of z that could possibly be in $Z(G)$ is the $p^{1+\lfloor \frac{c-2}{p-1} \rfloor}$ -st power.

If $\lfloor \frac{c-2}{p-1} \rfloor = \lfloor \frac{c-1}{p-1} \rfloor$, then we are done. So the only case that needs to be dealt with is the case considered in the previous section, when $c = 1 + (r-1)(p-1)$ for some positive integer $r > 1$.

Here we use the universal property of the coproduct. Let K be the group from Lemma 2.2. Since \mathcal{G} is the coproduct of C_p and \mathbf{Z} in \mathfrak{N}_{c+1} , the morphisms $C_p \rightarrow K$ given by $a \mapsto y$, and $\mathbf{Z} \rightarrow K$ given by $z \mapsto x_0 \in K$, induce a unique homomorphism $\varphi: \mathcal{G} \rightarrow K$. The image of $Z(\mathcal{G})$ must lie in $Z(K)$ (since the map is surjective). Since $\varphi(z^{p^{r-1}}) = x_0^{p^{r-1}}$ does not commute with y , we conclude that $z^{p^{r-1}} \notin Z(\mathcal{G})$. This proves that the smallest power of z that could lie in $Z(\mathcal{G})$ is z^{p^r} , which gives the desired inclusion. \square

Now let $G = \mathcal{G}/Z(\mathcal{G})$. This is a group of class c , minimally generated by $aZ(\mathcal{G})$ and $zZ(\mathcal{G})$. The former has order p , and the latter element has order $p^{1+\lfloor \frac{c-1}{p-1} \rfloor}$ by the proposition above. Thus G is a capable group of class c , minimally generated by two elements whose orders satisfy the equality in (1.1), showing that the inequality is indeed best possible.

Remark 3.3. I believe that in general inequality (1.1) will be both necessary and sufficient for the capability of a c -nilpotent product of cyclic p -groups. This is indeed the case when $c < p$ and when $p = c = 2$; see [5]. However, I have not been able to establish this for arbitrary p and c , which forced the somewhat indirect approach taken in this note.

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